Norm (mathematics)

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In linear algebra, functional analysis and related areas of mathematics, a **norm** is a function that assigns a strictly positive *length* or *size* to all vectors in a vector space, other than the zero vector. A **seminorm**, on the other hand, is allowed to assign zero length to some non-zero vectors.

A simple example is the 2-dimensional Euclidean space $\mathbf{R}^2$ equipped with the Euclidean norm. Elements in this vector space (e.g., (3, 7)) are usually drawn as arrows in a 2-dimensional cartesian coordinate system starting at the origin (0, 0). The Euclidean norm assigns to each vector the length of its arrow. Because of this, the Euclidean norm is often known as the magnitude.

A vector space with a norm is called a normed vector space. Similarly, a vector space with a seminorm is called a seminormed vector space.

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Definition

Given a vector space V over a subfield F of the complex numbers a **norm on** V is a function $p: V \to F$ with the following properties:[1]

For all $a \in F$ and all $\mathbf{u}, \mathbf{v} \in V$,

1. $p(a\mathbf{v}) = |a|\, p(\mathbf{v})$, (*positive homogeneity* or *positive scalability*).
2. $p(\mathbf{u} + \mathbf{v}) \le p(\mathbf{u}) + p(\mathbf{v})$ (*triangle inequality* or *subadditivity*).
3. If $p(\mathbf{v}) = 0$ then $\mathbf{v}$ is the zero vector (*separates points*).

A simple consequence of the first two axioms, positive homogeneity and the triangle inequality, is

$p(\mathbf{0}) = 0$ and thus

$p(\mathbf{v}) \geq 0$ (*positivity*).

A **seminorm** is a **norm** with the 3rd property (separating points) removed.

Although every vector space is seminormed (e.g., with the trivial seminorm in the Examples section below), it may not be normed. Every vector space V with seminorm $p(\mathbf{v})$ induces a normed space V/W, called the quotient space, where W is the subspace of V consisting of all vectors $\mathbf{v}$ in V with $p(\mathbf{v}) = 0$. The induced norm on V/W is clearly well-defined and is given by:

$p(W + \mathbf{v}) = p(\mathbf{v})$.

A topological vector space is called **normable** (**seminormable**) if the topology of the space can be induced by a norm (seminorm).

The norm of a vector $\mathbf{v}$ is usually denoted $\|\mathbf{v}\|$, and sometimes $|\mathbf{v}|$. However, the latter notation is generally discouraged, because it is also used to denote the absolute value of scalars and the determinant of matrices.

Examples

- All norms are seminorms.
- The *trivial seminorm*, with $p(\mathbf{x}) = 0$ for all $\mathbf{x}$ in V.
- The absolute value is a norm on the real numbers.
- Every linear form f on a vector space defines a seminorm by $\mathbf{x} \to |f(\mathbf{x})|$.

Euclidean norm

Main article: Euclidean distance

On an n-dimensional Euclidean space $\mathbf{R}^n$, the intuitive notion of length of the vector $\boldsymbol{x} = (x_1, x_2, ..., x_n)$ is captured by the formula

$$\|\boldsymbol{x}\| := \sqrt{x_1^2 + \cdots + x_n^2}.$$

This gives the ordinary distance from the origin to the point $\boldsymbol{x}$, a consequence of the Pythagorean theorem. The Euclidean norm is by far the most commonly used norm on $\mathbf{R}^n$, but there are other norms on this vector space as will be shown below. However all these norms are equivalent in the sense that they all define the same topology.

On an n-dimensional complex space $\mathbf{C}^n$ the most common norm is

$$\|\boldsymbol{z}\| := \sqrt{|z_1|^2 + \cdots + |z_n|^2} = \sqrt{z_1 \bar{z}_1 + \cdots + z_n \bar{z}_n}.$$

In both cases we can also express the norm as the square root of the inner product of the vector and itself:

$$\|\boldsymbol{x}\| := \sqrt{\boldsymbol{x}^* \boldsymbol{x}},$$

where $\boldsymbol{x}$ is represented as a column vector ($[x_1; x_2; ...; x_n]$), and $\boldsymbol{x}^*$ denotes its conjugate

transpose.

This formula is valid for any inner product space, including Euclidean and complex spaces. For Euclidean spaces, the inner product is equivalent to the dot product. Hence, in this specific case the formula can be also written with the following notation:

$$\|x\| := \sqrt{x \cdot x}.$$

The Euclidean norm is also called the **Euclidean length**, L^2 **distance**, ℓ^2 **distance**, L^2 **norm**, or ℓ^2 **norm**; see L^p space.

The set of vectors in $\mathbf{R}^{n+1}$ whose Euclidean norm is a given positive constant forms an n-sphere.

Euclidean norm of a complex number

The Euclidean norm of a complex number is the absolute value (also called the **modulus**) of it, if the complex plane is identified with the Euclidean plane $\mathbf{R}^2$. This identification of the complex number **x + iy** as a vector in the Euclidean plane, makes the quantity $\sqrt{x^2 + y^2}$ (as first suggested by Euler) the Euclidean norm associated with the complex number.

Taxicab norm or Manhattan norm

Main article: Taxicab geometry

$$\|x\|_1 := \sum_{i=1}^{n} |x_i|.$$

The name relates to the distance a taxi has to drive in a rectangular street grid to get from the origin to the point *x*.

The set of vectors whose 1-norm is a given constant forms the surface of a cross polytope of dimension equivalent to that of the norm minus 1. The Taxicab norm is also called the L_1 **norm**. The distance derived from this norm is called the Manhattan distance or L_1 **distance**.

In contrast,

$$\sum_{i=1}^{n} x_i$$

is not a norm because it may yield negative results.

p-norm

Main article: L^p *space*

Let *p* ≥ 1 be a real number.

$$\|\mathbf{x}\|_p := \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}.$$

Note that for *p* = 1 we get the taxicab norm, for *p* = 2 we get the Euclidean norm, and for *p* = ∞ we get the infinity norm or maximum norm.

This definition is still of some interest for 0 < *p* < 1, but the resulting function does not define a norm,[2] because it violates the triangle inequality. What is true, even in the measurable analog, is that the corresponding L^p class is a vector space, and it is also true that the function

$$\int_X |f(x) - g(x)|^p d\mu$$

(without *p*-th root) defines a distance that makes $L^p(X)$ into a complete metric topological vector space. These spaces are of great interest in functional analysis, probability theory, and harmonic analysis.

However, outside trivial cases, this topological vector space is not locally convex and has no continuous nonzero linear forms. Thus the topological dual space contains only the zero functional.

Maximum norm (special case of: infinity norm, uniform norm, or supremum norm)

Main article: Maximum norm

$$\|\mathbf{x}\|_\infty := \max(|x_1|, \ldots, |x_n|).$$

The set of vectors whose infinity norm is a given constant, *c*, forms the surface of a hypercube with edge length 2*c*.
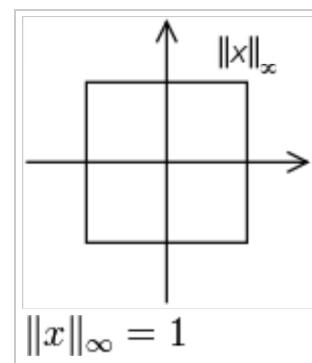
Zero norm

In probability and functional analysis, the zero norm induces a complete metric topology for the space of measureable functions and for the F-space of sequences with F–norm $(x_n) \mapsto \sum_n 2^{-n} x_n / (1 + x_n)$, which is discussed by Stefan Rolewicz in *Metric Linear Spaces*.[3]

Hamming distance of a vector from zero

See also: Hamming distance and discrete metric

In metric geometry, the discrete metric takes the value one for distinct points and zero otherwise. When applied coordinate-wise to the elements of a vector space, the discrete distance defines the *Hamming distance*, which is important in coding and information theory. In the field of real or complex numbers, the distance of the discrete metric from zero is not homogeneous in the non-zero point; indeed, the distance from zero remains one as its non-zero argument approaches zero. However, the discrete distance of a number from zero does satisfy the other properties of a norm, namely the triangle inequality and positive definiteness. When applied component-wise to vectors, the discrete distance from zero behaves like a non-homogeneous "norm", which counts the number of non-zero components in its vector argument; again, this non-homogeneous "norm" is discontinuous.

In signal processing and statistics, David Donoho referred to the *zero* "*norm*" with quotation marks. Following Donoho's notation, the zero "norm" of **x** is simply the number of non-zero

elements of **x**, or the Hamming distance of the vector from zero. When this "norm" is localized to a bounded set, it is the limit of p-norms as p approaches 0. Of course, the zero "norm" is **not** a B-norm, because it is not positive homogeneous. It is not even an F-norm, because it is discontinuous, jointly and severally, with respect to the scalar argument in scalar-vector multiplication and with respect to its vector argument. Abusing terminology, some engineers omit Donoho's quotation marks and inappropriately call the number-of-nonzeros function the L0 norm (sic.), also misusing the notation for the Lebesgue space of measurable functions.

Other norms

Other norms on $\mathbf{R}^n$ can be constructed by combining the above; for example

$$\|\boldsymbol{x}\| := 2|x_1| + \sqrt{3|x_2|^2 + \max(|x_3|, 2|x_4|)^2}$$

is a norm on $\mathbf{R}^4$.

For any norm and any injective linear transformation A we can define a new norm of x, equal to

$$\|Ax\|.$$

In 2D, with A a rotation by 45° and a suitable scaling, this changes the taxicab norm into the maximum norm. In 2D, each A applied to the taxicab norm, up to inversion and interchanging of axes, gives a different unit ball: a parallelogram of a particular shape, size and orientation. In 3D this is similar but different for the 1-norm (octahedrons) and the maximum norm (prisms with parallelogram base).

All the above formulas also yield norms on $\mathbf{C}^n$ without modification.

Infinite-dimensional case

The generalization of the above norms to an infinite number of components leads to the L^p spaces, with norms

$$\|x\|_p = \left(\sum_{i\in\mathbb{N}} |x_i|^p\right)^{1/p} \text{ resp. } \|f\|_{p,X} = \left(\int_X |f(x)|^p \, \mathrm{d}x\right)^{1/p}$$

(for complex-valued sequences x resp. functions f defined on $X \subset \mathbb{R}$), which can be further generalized (see Haar measure).

Any inner product induces in a natural way the norm $\|x\| := \sqrt{\langle x, x\rangle}$.

Other examples of infinite dimensional normed vector spaces can be found in the Banach space article.

Properties

The concept of unit circle (the set of all vectors of norm 1) is different in different norms: for the 1-norm the unit circle in $\mathbf{R}^2$ is a square, for the 2-norm (Euclidean norm) it is the well-known unit circle, while for the infinity norm it is a different square. For any p-norm it is a superellipse (with congruent axes). See the accompanying illustration. Note that due to the definition of the norm,

the unit circle is always convex and centrally symmetric (therefore, for example, the unit ball may be a rectangle but cannot be a triangle).

In terms of the vector space, the seminorm defines a topology on the space, and this is a Hausdorff topology precisely when the seminorm can distinguish between distinct vectors, which is again equivalent to the seminorm being a norm. The topology thus defined (by either a norm or a seminorm) can be understood either in terms of sequences or open sets. A sequence of vectors $\{v_n\}$ is said to converge in norm to v if $\|v_n - v\| \to 0$ as $n \to \infty$. Equivalently, the topology consists of all sets that can be represented as a union of open balls.

Two norms $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ on a vector space V are called *equivalent* if there exist positive real numbers C and D such that

$$C\|x\|_\alpha \le \|x\|_\beta \le D\|x\|_\alpha$$

for all x in V. For instance, on $\mathbf{C}^n$, if $p > r > 0$, then

$$\|x\|_p \le \|x\|_r \le n^{(1/r - 1/p)}\|x\|_p.$$

In particular,

$$\|x\|_2 \le \|x\|_1 \le \sqrt{n}\|x\|_2$$
$$\|x\|_\infty \le \|x\|_2 \le \sqrt{n}\|x\|_\infty$$
$$\|x\|_\infty \le \|x\|_1 \le n\|x\|_\infty.$$

Illustrations of unit circles in different norms.

If the vector space is a finite-dimensional real/complex one, all norms are equivalent. If not, some norms are not.

Equivalent norms define the same notions of continuity and convergence and for many purposes do not need to be distinguished. To be more precise the uniform structure defined by equivalent norms on the vector space is uniformly isomorphic.

Every (semi)-norm is a sublinear function, which implies that every norm is a convex function. As a result, finding a global optimum of a norm-based objective function is often tractable.

Given a finite family of seminorms p_i on a vector space the sum
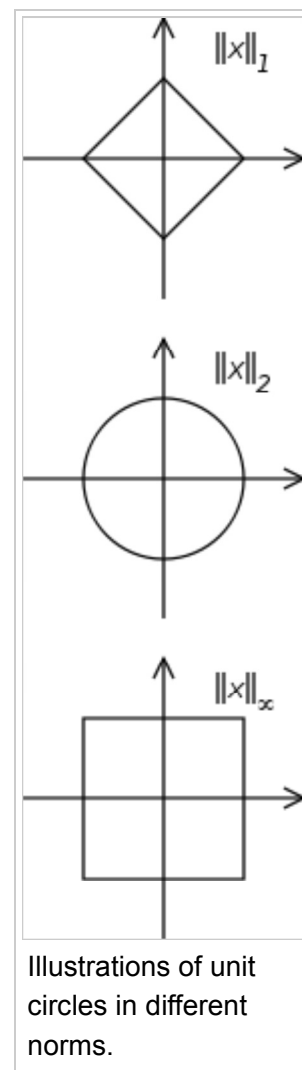
$$p(x) := \sum_{i=0}^{n} p_i(x)$$

is again a seminorm.

For any norm p on a vector space V, we have that for all **u** and **v** $\in$ V:

p(**u** ± **v**) ≥ | p(**u**) − p(**v**) |

For the l^p norms, we have the Hölder's inequality[4]

$$|x^\top y| \le \|x\|_p \|y\|_q \qquad \frac{1}{p} + \frac{1}{q} = 1.$$

A special case of the above property is the Cauchy–Schwarz inequality:[4]

$$|x^\top y| \leq \|x\|_2 \|y\|_2.$$

Classification of seminorms: Absolutely convex absorbing sets

All seminorms on a vector space V can be classified in terms of absolutely convex absorbing sets in V. To each such set, A, corresponds a seminorm p_A called the **gauge** of A, defined as

$p_A(x) := \inf\{\alpha : \alpha > 0, x \in \alpha A\}$

with the property that

$\{x : p_A(x) < 1\} \subseteq A \subseteq \{x : p_A(x) \leq 1\}$.

Conversely:

Any locally convex topological vector space has a local basis consisting of absolutely convex sets. A common method to construct such a basis is to use a separating family (p) of seminorms p: the collection of all finite intersections of sets {p<1/n} turns the space into a locally convex topological vector space so that every p is continuous.

A such method is used to design weak and weak* topologies.

norm case:

Suppose now that (p) contains a single p: since (p) is separating, p is a norm, and $A=\{p<1\}$ is its open unit ball. Then A is an absolutely convex bounded neighbourhood of 0, and $p = p_A$ is continuous.

The converse is due to Kolmogorov: any locally convex and locally bounded topological vector space is normable. Precisely:
If V is an absolutely convex bounded neighbourhood of 0, the gauge g_V (so that $V=\{g_V <1\}$) is a norm.

See also

- Asymmetric norm, a generalization of a norm for which ||x|| and ||−x|| are not necessarily equal.
- Matrix norm
- Mahalanobis distance
- Manhattan distance
- Relation of norms and metrics, a translation invariant and homogeneous metric can be used to define a norm.
- Normed vector space
- One-form

Notes

1. ^ Prugovečki 1981, page 20 (http://books.google.com/books?id=GxmQxn2PF3IC&pg=PA20)
2. ^ Except in $\mathbf{R}^1$, where it coincides with the Euclidean norm, and $\mathbf{R}^0$, where it is trivial.
3. ^ Rolewicz, Stefan (1987), *Functional analysis and control theory: Linear systems*, Mathematics and its Applications (East European Series), **29** (Translated from the Polish by Ewa Bednarczuk ed.), Dordrecht; Warsaw: D. Reidel Publishing Co.; PWN—Polish Scientific Publishers, pp. xvi+524, ISBN 9027721866, MR920371 (http://www.ams.org/mathscinet-getitem?mr=920371) , OCLC 13064804 (http://www.worldcat.org/oclc/13064804)
4. ^ *a b* Golub, Gene; Charles F. Van Loan (1996). *Matrix Computations - Third Edition*. Baltimore: The Johns Hopkins University Press. p. 53. ISBN 0-8018-5413-X.

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